Eletromagnetismo II - Prova 3

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16 de junho de 2025

Início: 14:00 - duração: 3:00 horas

Só serão consideradas as respostas que forem devidamente justificadas.

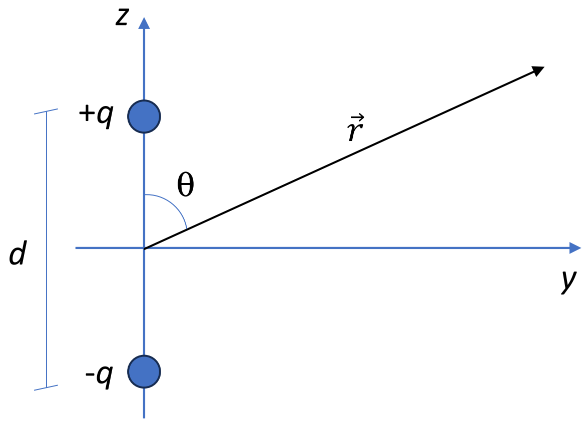
Não é permitido o uso de calculadoras.

Questão 01: (3,0) Potencial de um dipolo elétrico oscilante

Considere o dipolo elétrico apresentado na figura, onde as cargas positiva e negativa oscilam segundo a equação

$$q(t) = q_0 \cos(\omega t).$$

Há um fio condutor conectando as duas cargas. Calcule o potencial na posição $\vec{r}$ e no instante t. Lembre-se de levar em conta a velocidade finita c da propagação da informação. Não é necessário realizar nenhuma aproximação.



Questão 02: Radiação de dipolo elétrico

Conforme visto em sala de aula, os potenciais retardados de um dipolo elétrico oscilante, na zona de radiação e quando $d \ll r$ e $d \ll c/\omega$, valem

$$V(r,\theta,t) = -\frac{q_0 \omega d}{4\pi\epsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin\left[\omega(t - r/c)\right]$$

$$\vec{A}(r,\theta,t) = -\frac{\mu_0 q_0 \omega d}{4\pi r}\sin\left[\omega(t - r/c)\right]\hat{z}$$

(a) **(2,0)** Calcule o campo elétrico gerado pelo dipolo. Considere a zona de radiação, desprezando todos os termos que vão com $\sim 1/r^2$.

(b) **(2,0)** Calcule o campo magnético gerado pelo dipolo. Considere a zona de radiação, desprezando todos os termos que vão com $\sim 1/r$.

(c) **(2,0)** Mostre que a intensidade irradiada pelo dipolo vale

$$\left\langle \vec{S} \right\rangle = \left(\frac{\mu_0 q_0^2 d^2 \omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta}{r^2}\hat{r}$$

(d) **(1,0)** Mostre que a potência total irradiada vale

$$P = \frac{\mu_0 q_0^2 d^2 \omega^4}{12\pi c}$$

Equações que podem ser úteis:

$d\tau = r^2 \sin\theta dr d\theta d\phi$

$$\int \sin^3 x dx = -\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x$$

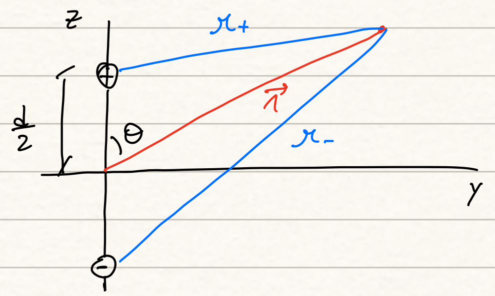
$$\vec{\nabla} V = \frac{\partial V}{\partial r}\,\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\,\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}\,\hat{\varphi}$$

$$\vec{\nabla}\times\vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\varphi \sin\theta) - \frac{\partial A_\theta}{\partial\varphi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial}{\partial r}(rA_\varphi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\varphi}$$

$\hat{z} = cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$

GABARITO

1- $q(t) = q_0 \cos(\omega t)$



$$V_+ = \frac{1}{4\pi\epsilon_0}\frac{q}{r_+}$$

$$r_+^2 = r^2 + d^2/4 - 2\frac{d}{2} r\cos\theta$$

$$\begin{cases} r_+ = \left(r^2 + d^2/4 - rd\cos\theta\right)^{1/2} \\ r_- = \left(r^2 + d^2/4 + rd\cos\theta\right)^{1/2} \\ q^{\pm}(t) = q_0 \cos\left[\omega\left(t - r_\pm/c\right)\right] \end{cases} \Rightarrow$$

$$V = V_+ + V_-$$

$$\boxed{V(r,t) = \frac{q_0}{4\pi\epsilon_0}\left[\frac{\cos\omega(t - r_+/c)}{r_+} - \frac{\cos\omega(t - r_-/c)}{r_-}\right], \quad \text{onde} \quad r_\pm = \left(r^2 + d^2/4 \mp rd\cos\theta\right)^{1/2}}$$

2- $V(r,\theta,t) = -\frac{q_0 \omega d}{4\pi\epsilon_0 c}\left(\frac{\cos\theta}{r}\right) \operatorname{sen}\omega(t - r/c)$

$\vec{A}(r,\theta,t) = -\frac{\mu_0 q_0 \omega d}{4\pi r} \operatorname{sen}\omega(t - r/c)\,\hat{z}$

A) $\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$

$$\vec{E} = \frac{q_0 \omega d}{4\pi\epsilon_0 c}\left\{\left[\cancel{-\frac{\cos\theta}{r^2}\operatorname{sen}\omega(t - r/c)}^{\,0} - \frac{\omega\cos\theta}{rc}\cos\omega(t - r/c)\right]\hat{r} + \left[\frac{1}{r}\cdot\cancel{\frac{1}{r}\operatorname{sen}\omega(t - r/c)\cdot(-\operatorname{sen}\theta)}^{\,0}\right]\hat{\theta}\right\}$$
$$+\frac{\mu_0 q_0 \omega d}{4\pi r}\cdot\omega\cos\omega(t - r/c)\,\hat{z}$$

$\hat{z} = \cos\theta\,\hat{r} - \operatorname{sen}\theta\,\hat{\theta}$; $c^2 = \frac{1}{\mu_0\epsilon_0} \Rightarrow \mu_0 = \frac{1}{c^2\epsilon_0} \Rightarrow$

$$\vec{E} = -\frac{\mu_0 q_0 \omega^2 d}{4\pi}\cancel{\frac{\cos\theta}{r}}\cos\omega(t - r/c)\,\hat{r} + \frac{\mu_0 q_0 \omega^2 d}{4\pi r}\cos\omega(t - r/c)\left(\cancel{\cos\theta\,\hat{r}} - \operatorname{sen}\theta\,\hat{\theta}\right)$$

$$\boxed{\vec{E} = -\frac{\mu_0 q_0 \omega^2 d}{4\pi}\left(\frac{\operatorname{sen}\theta}{r}\right)\cos\omega(t-r/c)\,\hat{\theta}}$$

B) $\vec{B} = \vec{\nabla}\times\vec{A}$

$$\operatorname{sen}\left(\omega t - \frac{\omega r}{c}\right)$$

$$\vec{B} = -\frac{\mu_0 q_0 \omega d}{4\pi}\,\vec{\nabla}\times\left[\frac{\operatorname{sen}\omega(t-r/c)\,\hat{z}}{r}\right]$$

$$\vec{B} = -\frac{\mu_0 q_0 \omega d}{4\pi}\,\vec{\nabla}\times\left[\frac{1}{r}\operatorname{sen}\omega(t-r/c)\cos\theta\,\hat{r} - \frac{1}{r}\operatorname{sen}\omega(t-r/c)\operatorname{sen}\theta\,\hat{\theta}\right]$$

$$\vec{B} = -\frac{\mu_0 q_0 \omega d}{4\pi}\,\frac{1}{r}\left\{\frac{\partial}{\partial r}\left[-\operatorname{sen}\omega(t-r/c)\operatorname{sen}\theta\right] - \frac{\partial}{\partial\theta}\left[\frac{1}{r}\operatorname{sen}\omega(t-r/c)\cos\theta\right]\right\}\hat{\phi}$$

$$\vec{B} = -\frac{\mu_0 q_0 \omega d}{4\pi r}\left[\frac{\omega}{c}\operatorname{sen}\theta\cos\omega(t-r/c) + \cancelto{0}{\frac{1}{r}\operatorname{sen}\theta\operatorname{sen}\omega(t-r/c)}\right]\hat{\phi}$$

$$\boxed{\vec{B} = -\frac{\mu_0 q_0 \omega^2 d}{4\pi c}\left(\frac{\operatorname{sen}\theta}{r}\right)\cos\omega(t-r/c)\,\hat{\phi}}$$

c) $\vec{S} = \dfrac{\vec{E}\times\vec{B}}{\mu_0}$

$$\vec{S} = \frac{\mu_0 q_0^2 \omega^4 d^2}{16\pi^2 c}\left(\frac{\operatorname{sen}\theta}{r}\right)^2\cos^2\omega(t-r/c)\,\hat{\theta}\times\hat{\phi}$$

$$\boxed{\langle\vec{S}\rangle_t = \frac{\mu_0 q_0^2 \omega^4 d^2}{32\pi^2 c}\,\frac{\operatorname{sen}^2\theta}{r^2}\,\hat{r}}$$

D) $P = \displaystyle\int\langle\vec{S}\rangle\cdot d\vec{a}$

$$P = \frac{\mu_0 q_0^2 \omega^4 d^2}{32\pi^2 c}\int\frac{\operatorname{sen}^2\theta}{r^2}\,r^2\operatorname{sen}\theta\,d\theta\,d\phi$$

$$= \frac{\mu_0 q_0^2 \omega^4 d^2}{32\pi^2 c}\cdot 2\pi\cdot\int_0^\pi \operatorname{sen}^3\theta\,d\theta$$

$$\frac{1}{16}\cdot\frac{4}{3} = \frac{1}{12}$$

$$\int_0^\pi \operatorname{sen}^3\theta\,d\theta = -\frac{1}{3}\operatorname{sen}^2\theta\cos\theta\Big|_0^\pi - \frac{2}{3}\cos\theta\Big|_0^\pi$$

$$= 0 - \frac{2}{3}\cdot(-1-1) = \frac{4}{3} \Rightarrow \boxed{P = \frac{\mu_0 q_0^2 \omega^4 d^2}{12\pi c}}$$